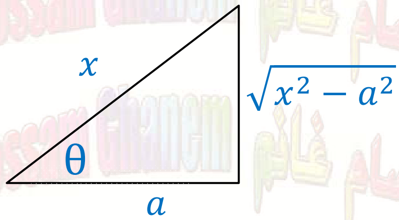
HOSSAM GHANEM

(22) 8.3 *Trigonometric Substitutions (B)*

Expression in integrand	$\sqrt{x^2 - a^2}$
Trigonometric substitution	$x = a \sec\theta \qquad dx = a \sec\theta \tan\theta \;\; d\theta$ $\sec\theta = \frac{x}{a}$ $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ 

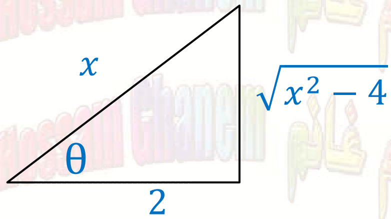
EXAMPLE

$$I = \int \frac{1}{\sqrt{x^2 - 4}}\, dx$$

$$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta \;\; d\theta$$

$$\sec\theta = \frac{x}{2}$$

$$\theta = \sec^{-1}\left(\frac{x}{2}\right)$$



$$I = \int \frac{1}{\sqrt{(2\sec\theta)^2 - 4}}\, 2\sec\theta\tan\theta \;\; d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}\; d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{2\sqrt{\sec^2\theta - 1}}\, d\theta = \int \frac{2\sec\theta\tan\theta}{2\sqrt{\tan^2\theta}}\; d\theta$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C$$

Notes

$I = \int \frac{x}{\sqrt{x^2+25}}\,dx$	$I = \int \frac{1}{\sqrt{x^2+25}}\,dx$
Let $t = x^2 + 2$ $\quad dt = 2x\,dx$ $I = \int \frac{x}{\sqrt{x^2+25}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt$	Let $x = 5\tan\theta$ $\quad dx = 5\sec^2\theta\ d\theta$ $I = \int \frac{1}{\sqrt{x^2+25}}\,dx = \int \frac{\sec^2\theta}{\sqrt{25\tan^2\theta+25}}\,dt$

Example 1

55 July 23 , 2011

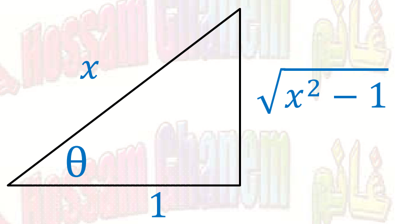
Evaluate the following integrals. $\int \frac{x^6+2}{x^3\sqrt{x^2-1}}\,dx$ (3 pts)

Solution

$x = \sec\theta \qquad dx = \sec\theta\tan\theta\ d\theta$

$\sec\theta = \frac{x}{1}$

$\theta = \sec^{-1} x$



$$I = \int \frac{x^6+2}{x^3\sqrt{x^2-1}}dx = \int \frac{\sec^6\theta+2}{\sec^3\theta\ \sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\ d\theta = \int \frac{\sec^6\theta+2}{\sec^3\theta\ \sqrt{\tan^2\theta}}\sec\theta\tan\theta\ d\theta$$

$$= \int \frac{\sec^6\theta+2}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\ d\theta = \int \frac{\sec^6\theta+2}{\sec^2\theta}d\theta$$

$$= \int \frac{\sec^6\theta}{\sec^2\theta} + \frac{2}{\sec^2\theta}d\theta = \int \sec^4\theta + 2\cos^2\theta\,d\theta = \int \sec^2\theta\sec^2\theta + 1 + \cos 2\theta\,d\theta$$

$$= \int (1+\tan^2\theta)\sec^2\theta + 1 + \cos 2\theta\,d\theta$$

$$I_1 = \int (1+\tan^2\theta)\sec^2\theta\,d\theta$$

Let $\quad u = \tan\theta \qquad du = \sec^2\theta\,d\theta$

$$I_1 = \int (1+u^2)du = u + \frac{1}{3}u + c_1 = \tan\theta + \frac{1}{3}\tan^3\theta + c_1$$

$$I_2 = \int 1 + \cos 2\theta\,d\theta = \theta + \frac{1}{2}\sin 2\theta + c_2$$

$$I = \tan\theta + \frac{1}{3}\tan^3\theta + \theta + \frac{1}{2}\sin 2\theta + c = \tan\theta + \frac{1}{3}\tan^3\theta + \theta + \sin\theta\cos\theta + c$$

$$I = \sqrt{x^2-1} + \frac{1}{3}\left(\sqrt{x^2-1}\right)^3 + \sec^{-1}x + \frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{x} + c$$

Example 2

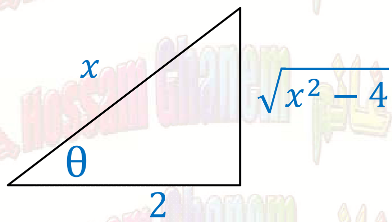
Evaluate the integral $\int x^3(x^2-4)^{\frac{3}{2}}\,dx$

Solution

$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\ d\theta$

$\sec\theta = \frac{x}{2}$

$\theta = \sec^{-1}\left(\frac{x}{2}\right)$



$$I = \int x^3(x^2-4)^{\frac{3}{2}}\,dx = \int 2^3\sec^3\theta\,(4\sec^2\theta-4)^{\frac{3}{2}}\cdot 2\sec\theta\tan\theta\ d\theta$$

$$= \int 2^3\sec^3\theta\,(4)^{\frac{3}{2}}(\tan^2\theta)^{\frac{3}{2}}\cdot 2\sec\theta\tan\theta\ d\theta = \int 2^3\sec^3\theta\,(2)^3\tan^3\theta\ \cdot 2\sec\theta\tan\theta\ d\theta$$

$$= (2)^7\int \sec^4\theta\tan^4\theta\ \ d\theta = (2)^7\int \sec^2\theta\tan^4\theta\cdot\sec^2\theta\ \ d\theta = (2)^7\int (1+\tan^2\theta)\tan^4\theta\cdot\sec^2\theta\ \ d\theta$$

Let $\quad t = \tan\theta \qquad dt = \sec^2\theta\ d\theta$

$$I = (2)^7\int (1+t^2)t^4\ \ dt = (2)^7\int (t^4+t^6)\ \ dt = (2)^7\left(\frac{1}{5}t^5+\frac{1}{6}t^6\right)+c$$

$$I = (2)^7\left(\frac{1}{5}\tan^5\theta+\frac{1}{6}\tan^6\theta\right)+c = (2)^7\left[\frac{1}{5}\left(\frac{\sqrt{x^2-4}}{2}\right)^5+\frac{1}{6}\left(\frac{\sqrt{x^2-4}}{2}\right)^6\right]+c$$

Example 3

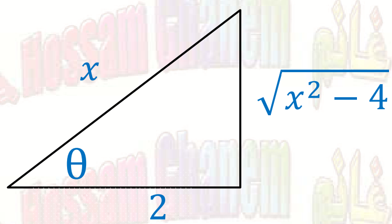
Evaluate the integral $\int \frac{\sqrt{x^2-4}}{x}\,dx$

Solution

$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\ d\theta$

$\sec\theta = \frac{x}{2}$

$\theta = \sec^{-1}\left(\frac{x}{2}\right)$



$$I = \int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\cdot\ 2\sec\theta\tan\theta\ d\theta$$

$$= \int \frac{2\sqrt{\sec^2\theta-1}}{2\sec\theta}\cdot\ 2\sec\theta\tan\theta\ d\theta = \int \frac{2\tan\theta}{2\sec\theta}\cdot\ 2\sec\theta\tan\theta\ d\theta$$

$$= 2\int \tan^2\theta\ d\theta\ = 2\int (\sec^2\theta-1)\,d\theta = 2\tan\theta - 2\theta + c = 2\ \frac{\sqrt{x^2-4}}{2} - 2\sec^{-1}\left(\frac{x}{2}\right)+c$$

$$= \sqrt{x^2-4}\ - 2\sec^{-1}\left(\frac{x}{2}\right)+c$$

Example 4 Evaluate the integral $\int \frac{\sinh x}{\sqrt{\sinh^2 x - 3}}\, dx$ 35 December 2004

Solution

$$I = \int \frac{\sinh x}{\sqrt{\sinh^2 x - 3}}\, dx = \int \frac{\sinh x}{\sqrt{\cosh^2 x - 4}}\, dx$$

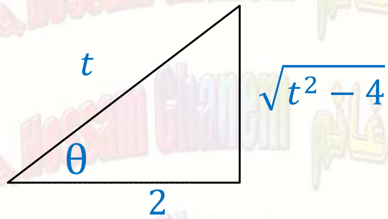
Let $t = \cosh x$ $\qquad dt = \sinh x\, dx$

$$I = \int \frac{\sinh x}{\sqrt{\cosh^2 x - 4}}\, dx = \int \frac{1}{\sqrt{t^2 - 4}}\, dt$$

$t = 2\sec\theta \qquad dt = 2\sec\theta\tan\theta\, d\theta$

$$\sec\theta = \frac{t}{2}$$

$$\theta = \sec^{-1}\left(\frac{t}{2}\right)$$



$$I = \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}\, d\theta = \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\, d\theta = 2\int \sec\theta\, d\theta = 2\ln|\sec\theta + \tan\theta| + c$$

$$= 2\ln\left|\frac{t}{2} + \frac{\sqrt{t^2 - 4}}{2}\right| + c = \ln\left|\frac{\cosh x}{2} + \frac{\sqrt{\cosh^2 x - 4}}{2}\right| + c$$

TO FIND $\int \frac{1}{m\sin x + n\cos x + C}\, dx$

استخدم التعويض التالي

$\sin x = \frac{2u}{1+u^2}$	$\cos x = \frac{1-u^2}{1+u^2}$	$\tan x = \frac{2u}{1-u^2}$	$dx = \frac{2}{1+u^2}\, du$	$u = \tan\frac{x}{2}$

Example 5 Evaluate the integral $\int \frac{1}{1 + \sin x + \cos x}\, dx$ 6 July 1996, 31 December 2003

Solution

$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad \tan x = \frac{2u}{1-u^2} \qquad dx = \frac{2}{1+u^2}\, du \qquad u = \tan\frac{x}{2}$

$$I = \int \frac{1}{1 + \sin x + \cos x}\, dx = \int \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{1 + u^2 + 2u + 1 - u^2}\, du = \int \frac{2}{2u+2}\, du = \int \frac{1}{u+1}\, du = \ln|u+1| + c = \ln\left|\tan\frac{x}{2} + 1\right| + c$$

Example 6 Evaluate the integral $\int \frac{1}{\tan x + \sin x} dx$ 8 May 1997, 15 December 1998

Solution

$\sin x = \frac{2u}{1+u^2}$ $\cos x = \frac{1-u^2}{1+u^2}$ $\tan x = \frac{2u}{1-u^2}$ $dx = \frac{2}{1+u^2} du$ $u = \tan\frac{x}{2}$

$$I = \int \frac{1}{\tan x + \sin x} dx = \int \frac{1}{\frac{2u}{1-u^2} + \frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{(1-u^2)(1+u^2)}{2u(1+u^2) + 2u(1-u^2)} \cdot \frac{2}{1+u^2} du$$

$$= \int \frac{1-u^2}{2u[1+u^2+1-u^2]} \cdot \frac{2}{1} du = \int \frac{1-u^2}{2u[2]} \cdot \frac{2}{1} du = \int \frac{1-u^2}{2u} du$$

$$= \frac{1}{2}\int \left(\frac{1}{u} - \frac{u^2}{u}\right) du = \frac{1}{2}\int \left(\frac{1}{u} - u\right) du = \frac{1}{2}\ln u - \frac{1}{4}u^2 = \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right| - \frac{1}{4}\left(\tan\left(\frac{x}{2}\right)\right)^2 + c$$

Example 7 Evaluate $\int \frac{2 + \tan\frac{x}{2}}{2\sin x + 2\cos x + 3} dx$ 35 December 2004, 41 July 2006

Solution

$\sin x = \frac{2u}{1+u^2}$ $\cos x = \frac{1-u^2}{1+u^2}$ $\tan x = \frac{2u}{1-u^2}$ $dx = \frac{2}{1+u^2} du$ $u = \tan\frac{x}{2}$

$$I = \int \frac{2+u}{2\cdot\frac{2u}{1+u^2} + 2\cdot\frac{1-u^2}{1+u^2} + 3} \cdot \frac{2}{1+u^2} du = \int \frac{2\cdot(2+u)}{4u + 2(1-u^2) + 3 + 3(1+u^2)} du$$

$$= \int \frac{2\cdot(2+u)}{4u + 2 - 2u^2 + 3 + 3u^2} du = \int \frac{2\cdot(2+u)}{u^2+4u+5} du = \int \frac{2u+4}{u^2+4u+5} du$$

$$= \ln|u^2+4u+5| + c = \ln\left|\tan^2\left(\frac{x}{2}\right) + 4\tan\left(\frac{x}{2}\right) + 5\right| + c$$

Homework

1	Evaluate	$\int \frac{1}{x\sqrt{x^4-9}}\,dx$	
2	Evaluate	$\int \frac{1}{\sqrt{x^2-16}}\,dx$	20 April 2000
3	Evaluate	$\int x^3(x^2-1)^{\frac{3}{2}}\,dx$	39 December 2005
4	Evaluate	$\int \frac{1}{2+\cos x+2\sin x}\,dx$	7 November 1996
5	Evaluate	$\int \frac{1}{\tan x+\sin x}\,dx$	42 December 2006
6	Evaluate	$\int \frac{1}{1+\sin 3x+\cos 3x}\,dx$	28 May 2003
7	Evaluate	$\int \frac{1}{3+\cos x-2\sin x}\,dx$	33 May 2004
8	Evaluate	$\int \frac{1}{4+\sin x}\,dx$	13 May 1998
9	Evaluate	$\int \frac{1}{8-4\sin x+7\cos x}\,dx$	12 December 1997
10	Evaluate	$\int \frac{1}{2+\sin x+2\cos x}\,dx$	14 November 1998
11	Evaluate	$\int \frac{1}{\sin x-\cos x+1}\,dx$	16 May 1999
12	Evaluate	$\int \frac{1}{1-\sin x-\cos x}\,dx$	34 July 2004
13	Evaluate	$\int \frac{1}{3-\sin x}\,dx$	37 June 2005
14	Evaluate	$\int \frac{1}{2\cos x+\sin x-2}\,dx$	38 July 2005
15	Evaluate	$\int \frac{\sec x}{3\tan x-4}\,dx$	26 July 2002

Homework

16	Evaluate the following integral : ($3\frac{1}{2}$ points)	$\int \frac{10\cos x + 10}{2\cos x - \sin x + 2}\,dx$	50 Dec. 15, 2009
17	Evaluate the following. [3.5 pts.]	$\int \frac{x^3}{\sqrt{x^2-1}}\,dx$	51 May 13, 2010
18	Evaluate the following. [3.5 pts.]	$\int \frac{1}{5-3\cos x}\,dx$	51 May 13, 2010
25	Evaluate the following integral [3 marks e]	$\int \frac{\csc x}{2\csc x + 1}\,dx$	52 July 24, 2010
26	(3 pts.) Evaluate the following integral	$\int \frac{\csc x}{(1+\cos x)^2}\,dx$	53 11 Dec. 2010
27	Evaluate the following integral	$\int \frac{1-\sin x}{(x^2+2x\cos x-\sin^2 x)^{3/2}}\,dx$	37 August 7, 2010
28	Evaluate the following integrals. (3 pts)	$\int \frac{\cos x}{\sin x + \cos x + 1}\,dx$	55 July 23 , 2011

*	Evaluate	$\int \frac{1}{1+2\sec x}\,dx$	44 July 2007

Solution

$$I = \int \frac{1}{1+2\sec x}\,dx = \int \frac{\cos x}{\cos x + 2}\,dx = \int \frac{\cos x + 2 - 2}{\cos x + 2}\,dx = \int \frac{\cos x + 2}{\cos x + 2} - \frac{2}{\cos x + 2}\,dx$$

$$= \int \left(1 - \frac{2}{\cos x + 2}\right) dx$$

$$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad \tan x = \frac{2u}{1-u^2} \qquad dx = \frac{2}{1+u^2}du \qquad u = \tan\frac{x}{2}$$

$$I = \int \left(1 - \frac{2}{\frac{1-u^2}{1+u^2} + 2}\right) \cdot \frac{2}{1+u^2}\,du = \int \left(\frac{2}{1+u^2} - \frac{4}{1-u^2+2+2u^2}\right) du = \int \left(\frac{2}{1+u^2} - \frac{4}{u^2+3}\right) du$$

$$= 2\tan^{-1} u - \frac{4}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c = 2\tan^{-1}\left(\tan\frac{x}{2}\right) - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{2}\right)\right) + c$$